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Difference equations for the correlation functions of the eight-vertex model

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Abstract. We propose that the correlation functions of the inhomogeneous eight-vertex model in the anti-ferroelectric regime satisfy a system of difference equations with respect to the spectral parameters. Solving the simplest difference equation we obtain the expression for the spontaneous staggered polarization conjectured by Baxter and Kelland. We also discuss a related construction of vertex operators on the lattice.

1. Introduction

In [1, 2] it was recognized that the correlation functions of the inhomogeneous six-vertex model in the anti-ferroelectric regime can be expressed as a trace of products of the q-deformed vertex operators. An explicit integral formula is given in [2]. These correlators satisfy a system of q-difference equations [3], that were introduced by Smirnov in the study of form factors in massive integrable QFT [4] and correspond to the level-0 case of the q-KZ equation of Frenkel and Reshetikhin [5]. (To be precise the equations for the correlators are 'dual' to Smirnov's ones, but we do not go into such details here). As remarked in [5], if one replaces the trigonometric R matrices appearing here by the elliptic ones, the resulting equations are still 'completely integrable'. In this paper we propose that the latter are precisely those satisfied by the correlators of the eight-vertex model in the anti-ferroelectric regime.

Our method is based simply on the Yang–Baxter equation and the crossing symmetry and, as such, is applicable to more general models. This construction also allows one to interpret the q-vertex operator employed in [1, 2] as an operator that inserts a dislocation (an extra half-infinite line) on the lattice. In this formulation the vertex operators generalize straightforwardly to the elliptic case. It remains an interesting open problem to give a mathematical construction of such operators, along with the relevant elliptic deformation of Lie algebras.

The text is organized as follows. In section 1, we introduce our notation, and formulate the properties of the general correlators including Smirnov's difference equations. We give a heuristic argument for derivation of the main statements. In section 2 we solve the simplest difference equation and derive the spontaneous staggered polarization. We obtain the expression conjectured by Baxter and Kelland [6]. Section 3 is devoted to the construction of vertex operators on the lattice.

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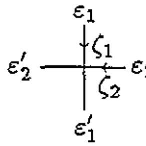
2. Correlators and Smirnov's equations

2.1. The eight-vertex model

Consider an infinite square lattice consisting of oriented lines, each carrying a spectral parameter varying from line to line. The orientation of each line will be shown by an arrow on it. A vertex is a crossing of two lines with spectral parameters, say ζ_1 and ζ_2, together with the adjacent four edges belonging to the crossing lines, as shown in (1). The edges are assigned state variables (which we call 'spins') : $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$. In the eight-vertex model, each spin can take one of the two different values $\pm$. A spin configuration around the vertex is an assignment of $\pm$ on the four edges. Notice that our notation is somewhat different from the standard. In particular, arrows on the lines denote the orientations that we assign to them, rather than the spins. There are 16 possible vertex configurations. We assign each configuration a Boltzmann weight. The set of all Boltzmann weights form the elements of the R-matrix:



$$\varepsilon_2 = R(\zeta_1/\zeta_2)_{\varepsilon_1'\varepsilon_2';\varepsilon_1\varepsilon_2}. \tag{1}$$

The matrix $R(\zeta)$ acts on $\mathbf{C}^2 \otimes \mathbf{C}^2$ via the natural basis $\{v_+, v_-\}$ of $\mathbf{C}^2$ as $R(\zeta)v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} = \sum v_{\varepsilon_1} \otimes v_{\varepsilon_2} R(\zeta)_{\varepsilon_1\varepsilon_2;\varepsilon_1'\varepsilon_2'}$. Normalized by the partition function per site κ, it is given explicitly by

$$R(\zeta) = \frac{1}{\kappa(\zeta)} \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array} \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a(\zeta) & 0 & 0 & d(\zeta) \\ 0 & b(\zeta) & c(\zeta) & 0 \\ 0 & c(\zeta) & b(\zeta) & 0 \\ d(\zeta) & 0 & 0 & a(\zeta) \end{pmatrix} \end{array}$$

where the unnormalized Boltzmann weights a, b, c, d (see (10.4.23–24) and (10.7.9) in [7]) are given by

$$a(\zeta) = -\mathrm{i}\rho\Theta(\mathrm{i}\lambda)H(\mathrm{i}\lambda - \mathrm{i}u)\Theta(\mathrm{i}u) \qquad b(\zeta) = -\mathrm{i}\rho\Theta(\mathrm{i}\lambda)\Theta(\mathrm{i}\lambda - \mathrm{i}u)H(\mathrm{i}u)$$
$$c(\zeta) = -\mathrm{i}\rho H(\mathrm{i}\lambda)\Theta(\mathrm{i}\lambda - \mathrm{i}u)\Theta(\mathrm{i}u) \qquad d(\zeta) = \mathrm{i}\rho H(\mathrm{i}\lambda)H(\mathrm{i}\lambda - \mathrm{i}u)H(\mathrm{i}u)$$

where we set $q = \exp(-\pi I'/I)$, $x = \exp(-\pi\lambda/2I)$, $\zeta = \exp(\pi u/2I)$. We shall restrict our discussion to the principal regime

$$0 < \sqrt{q} < x < \zeta^{-1} < 1$$

in which $\kappa(\zeta)$ is given by (see (10.8.44) in [7])

$$\kappa(\zeta) = \rho\gamma x^{-1}\bar{\kappa}(\zeta^2)(x^2\zeta^2; q)_\infty(qx^{-2}\zeta^{-2}; q)_\infty \qquad \gamma = q^{1/4}(q;q)_\infty^2(q^2;q^2)_\infty$$
$$\bar{\kappa}(z) = \frac{(x^4 z; q, x^4)_\infty (x^2 z^{-1}; q, x^4)_\infty (qz; q, x^4)_\infty (qx^2 z^{-1}; q, x^4)_\infty}{(x^4 z^{-1}; q, x^4)_\infty (x^2 z; q, x^4)_\infty (qz^{-1}; q, x^4)_\infty (qx^2 z; q, x^4)_\infty}.$$

The symbol $(z; q_1, \ldots, q_k)_\infty$ means $\prod_{n_1,\ldots,n_k=0}^{\infty}(1-zq_1^{n_1}\ldots q_k^{n_k})$. Notice that in the definition of the R-matrix, κ was used to normalize the Boltzmann weights.

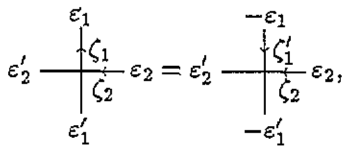
The R-matrix satisfies

$$R(\zeta)PR(\zeta^{-1})P = 1 \qquad \left(PR(\zeta^{-1})P\right)^{t_1} = (\sigma^x \otimes 1)R(\zeta/x)(\sigma^x \otimes 1)^{-1}. \tag{2}$$

Here $Pv \otimes v' = v' \otimes v$, $(\cdot)^{t_1}$ means the transpose with respect to the first component, and $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that

$$R(x^{-1}) = \begin{pmatrix} 0 & & & 0 \\ & 1 & 1 & \\ & 1 & 1 & \\ 0 & & & 0 \end{pmatrix} \tag{3}$$

and that (2) is equivalent to



$\zeta_1' = \zeta_1/x.$

Namely, reversing the orientation of a vertical line in (1) gives rise to reversing the spins on that line and shifting the spectral parameter $\zeta_1 \to \zeta_1' = \zeta_1/x$.

2.2. *Correlators and dislocations on the lattice*

Let us consider an infinite square lattice [8] with all the vertical lines oriented downward, all the horizontal lines to the left. We associate the spectral parameters ζ_j to the vertical lines, and ξ_k to the horizontal lines. As argued in [8], the calculation of correlators of arbitrary spins reduces to calculating correlators of the vertical-edge spins located on the same row, where by 'row' we mean the set of all vertical edges between two neighbouring horizontal lines. We shall restrict our attention to this case.

In the principal regime we are considering, the Boltzmann weight c dominates the others. In the low-temperature limit: $q, x, \xi_k/\zeta_j \to 0$, only type-$c$ vertex-configurations are non-vanishing. Therefore, the spin variables take the same value in the NE–SW direction, and alternate in the NW–SE direction. In this limit, two spin configurations are possible: the ground-state configurations (GSCs). At finite but low temperatures, we choose and fix a GSC, and consider the statistical sum over all configurations which differ by a finite number of spins from that GSC. We let $i \in \mathbf{Z}_2$ label the choice of GSC by the fact that in the low-temperature limit the spin on some reference edge is frozen to the value $(-1)^{i+1}$. The choice of the reference edge will be given later. Now choose a particular row, and consider n successive vertical edges on it. Let $\zeta_1, \ldots, \zeta_n$ be the spectral parameters attached to the corresponding lines, numbered from left to right. As the reference edge we take the next left to the one with the spectral parameter ζ_1. Consider the probability of the spins taking the values $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm\}$. Baxter's conclusion [8] was that this correlator is independent of all other spectral parameters.

In this work, we consider sums over all spin configurations on various lattices, with certain spins kept fixed. On infinite lattices, such configuration sums are not well defined. Only the ratios of such sums, on the same lattice but with the different choices of fixed spins, are well defined in low-temperature series expansions. When considering probabilities, the normalization is chosen so that the sum of all correlators is equal to 1. However, except for the simplest situation such as (4), such a choice does not give solutions to our system of equations. In fact, we do not know how to normalize our correlators. We postulate that there exists a proper normalization such that they satisfy all the required properties given below.

In order to obtain expressions that satisfy our difference equations we consider more general correlators than the probabilities of spins. We recover the original spin correlators by specializations (see (4)). We proceed as follows: we break the n lines at the chosen row, and change the spectral parameters of the lower halves to $\zeta_1', \ldots, \zeta_n'$. Then we consider the configuration sum with the $2n$ spins fixed to the values $\varepsilon_1, \ldots, \varepsilon_n$ from left to right on the upper row, and $\varepsilon_1', \ldots, \varepsilon_n'$ on the lower (see figure 1). We also consider a 'dislocated' lattice obtained as follows. Delete the n lower half-lines with the spectral parameters $\zeta_1', \ldots, \zeta_n'$. Therefore, we are left with the n upper half-lines whose spectral parameters are $\zeta_1, \ldots, \zeta_n$. Rename them to $\xi_{n+1}, \ldots, \xi_{2n}$. Now insert another set of n upper half-lines with spectral parameters $\xi_1, \ldots, \xi_n$ to the left of those which already exist. The orientation of the inserted lines is downward. Thus we obtain $2n$ half-lines with the spectral parameters $\xi_1, \ldots, \xi_{2n}$ from left to right (see figure 2). We consider the correlator of the $2n$ spins $\tau_1, \ldots, \tau_{2n}$ on the bottom edges of these half-lines, and denote it by $F_{2n}^{(i)}(\xi_1, \ldots, \xi_{2n})_{\tau_1,\ldots,\tau_{2n}}$.

One of our postulates is that the correlator of figure 1 is given by

$$F_{2n}^{(i+n)}(\zeta_n'/x, \ldots, \zeta_1'/x, \zeta_1, \ldots, \zeta_n)_{-\varepsilon_n',\ldots,-\varepsilon_1',\varepsilon_1,\ldots,\varepsilon_n}.$$

In section 1.3 we will give more general statements than this—see (11) and (12). If we restrict the variables in the above expression as follows:

$$F_{2n}^{(i+n)}(\zeta_n/x, \ldots, \zeta_1/x, \zeta_1, \ldots, \zeta_n)_{-\varepsilon_n,\ldots,-\varepsilon_1,\varepsilon_1,\ldots,\varepsilon_n} \tag{4}$$

we obtain the probability of the original n spins taking the values $\varepsilon_1, \ldots, \varepsilon_n$ on the regular infinite lattice (see (9)).

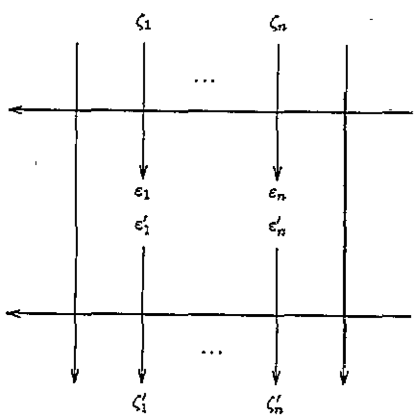


Figure 1. Breaking the successive n vertical lines.

Hereafter we will write $n, \zeta_j, \varepsilon_j$ for $2n, \xi_j, \tau_j$ in $F^{(i)}$, with the understanding that if n is odd, the correlator is zero. Set

$$F_n^{(i)}(\zeta_1, \ldots, \zeta_n) = \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm} F_n^{(i)}(\zeta_1, \ldots, \zeta_n)_{\varepsilon_1,\ldots,\varepsilon_n} v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n} \in \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2.$$

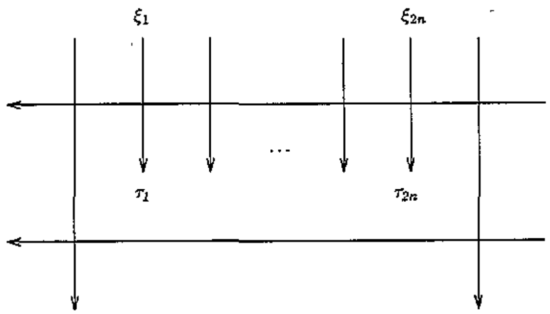


Figure 2. The dislocated lattice with $2n$ upper half-lines.

We will denote by $R_{jk}(\zeta)$ $(j < k)$ the matrix $R(\zeta)$ acting on the jth and the kth tensor components. We also use the transposition $P_{jk} = R_{jk}(1)$.

The following are the main results of this paper. In section 1.3 we will give a heuristic proof by considering $F_n^{(i)}(\zeta_1, \ldots, \zeta_n)_{\varepsilon_1,\ldots,\varepsilon_n}$ as a sum over all configurations with the spins $\varepsilon_1, \ldots, \varepsilon_n$ kept fixed.

Difference equation:

$$F_n^{(i)}(\zeta_1, \ldots, x^2\zeta_j, \ldots, \zeta_n) = R_{j\,j+1}(x^2\zeta_j/\zeta_{j+1})^{-1} \cdots R_{jn}(x^2\zeta_j/\zeta_n)^{-1} \times R_{1j}(\zeta_1/\zeta_j) \cdots R_{j-1\,j}(\zeta_{j-1}/\zeta_j) F_n^{(i+1)}(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n). \tag{5}$$

R-matrix symmetry:

$$F_n^{(i)}(\ldots, \zeta_{j+1}, \zeta_j, \ldots) = P_{j\,j+1} R_{j\,j+1}(\zeta_j/\zeta_{j+1}) F_n^{(i)}(\ldots, \zeta_j, \zeta_{j+1}, \ldots). \tag{6}$$

$\mathbf{Z}_2$-invariance and Parity:

$$F_n^{(i+1)}(\zeta_1, \ldots, \zeta_n)_{\varepsilon_1,\ldots,\varepsilon_n} = F_n^{(i)}(\zeta_1, \ldots, \zeta_n)_{-\varepsilon_1,\ldots,-\varepsilon_n} \tag{7}$$

$$F_n^{(i)}(\ldots, -\zeta_j, \ldots)_{\ldots,\varepsilon_j,\ldots} = (-)^{i+j-1}\varepsilon_j F_n^{(i)}(\ldots, \zeta_j, \ldots)_{\ldots,\varepsilon_j,\ldots}. \tag{8}$$

Normalization:

$$\sum_{\varepsilon=\pm} F_{n+2}^{(i)}(\ldots, \zeta_{j-1}, \zeta, x\zeta, \zeta_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon,-\varepsilon,\varepsilon_j,\ldots} = F_n^{(i)}(\ldots, \zeta_{j-1}, \zeta_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon_j,\ldots} \tag{9}$$

$$F_{n+2}^{(i)}(\ldots, \zeta_{j-1}, x\zeta, \zeta, \zeta_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon,\varepsilon',\varepsilon_j,\ldots} = \delta_{\varepsilon,-\varepsilon'} F_n^{(i)}(\ldots, \zeta_{j-1}, \zeta_j, \ldots)_{\ldots,\varepsilon_{j-1},\varepsilon_j,\ldots}. \tag{10}$$

2.3. Rotating a half-line

Let us consider the following more general correlators: choose a face of the lattice. Let $n_1 + n_2 = n$ (even). Insert n_1 upper half-lines and n_2 lower half-lines into the face, all of which are oriented downward. Note that in section 1.2 we considered the cases where $n_1 = n_2$ or $n_2 = 0$. We let the edge on the west side of the said face to be the reference edge used to label the sectors, i.e. in the low-temperature limit the spin on that edges is $(-1)^{i+1}$ (see figure 6). Denote the correlator of this dislocated lattice by $F^{(i)}(\zeta_1, \ldots, \zeta_{n_1}; \zeta'_1, \ldots, \zeta'_{n_2})_{\varepsilon_1,\ldots,\varepsilon_{n_1};\varepsilon'_1,\ldots,\varepsilon'_{n_2}}$. Here we place the spectral parameters and the spins of the upper lines first, from left to right, and then the lower, also from left to right. A clue to the difference equations is the following.

Rotation:

$$F^{(i)}(\zeta_1, \ldots, \zeta_{n_1}; \zeta'_1, \ldots, \zeta'_{n_2})_{\varepsilon_1,\ldots,\varepsilon_{n_1};\varepsilon'_1,\ldots,\varepsilon'_{n_2}}$$

$$= F^{(i+1)}(\zeta'_1/x, \zeta_1, \ldots, \zeta_{n_1}; \zeta'_2, \ldots, \zeta'_{n_2})_{-\varepsilon'_1,\varepsilon_1,\ldots,\varepsilon_{n_1};\varepsilon'_2,\ldots,\varepsilon'_{n_2}} \tag{11}$$

$$= F^{(i)}(\zeta_1, \ldots, \zeta_{n_1-1}; \zeta'_1, \ldots, \zeta'_{n_2}, \zeta_{n_1}/x)_{\varepsilon_1,\ldots,\varepsilon_{n_1-1};\varepsilon'_1,\ldots,\varepsilon'_{n_2},-\varepsilon_{n_1}}\,. \tag{12}$$

Let us give heuristic arguments to derive (5)–(12). We suppose that the correlators are given as configuration sums on the infinite lattice. It makes sense to do that because of the normalization of the Boltzmann weights by κ. We further assume that the vertices on the lattice that are pushed away to infinity by a Z-invariant deformation of the lattice can be neglected. Let us use this assumption in a concrete situation. Consider a part of the lattice as given in figure 3, on the left. The rest of the lattice, not shown in this figure, is irrelevant in the following argument. Denote the configuration sum with the spins ε_1 and ε_2 by $F(\varepsilon_1, \varepsilon_2; \Lambda)$. The configuration sum for the lattice on the right, is given by

$$F^{(i)}(\varepsilon_1, \varepsilon_2; \Lambda') = \sum_{\varepsilon'_1,\varepsilon'_2} R(\zeta_1/\zeta_2)_{\varepsilon_1,\varepsilon_2;\varepsilon'_1,\varepsilon'_2} F^{(i)}(\varepsilon'_1, \varepsilon'_2; \Lambda)\,. \tag{13}$$

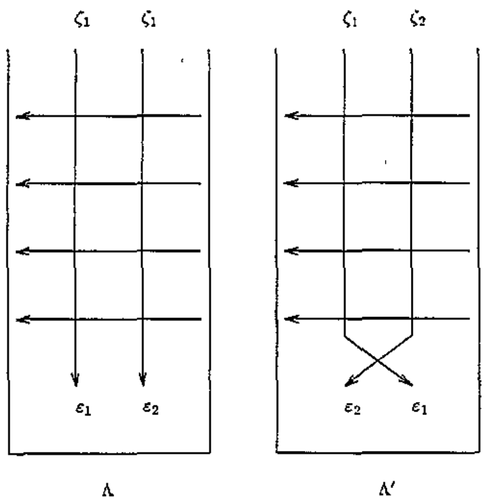


Figure 3. Twisting two vertical lines. The extra vertex in Λ' can be pushed up to infinity.

We can push out the extra vertex in Λ' to the north as far as we wish by using the Yang–Baxter equation, without changing the statistical sum $F^{(i)}(\varepsilon_1, \varepsilon_2; \Lambda')$. Therefore by the above assumption, we have $F^{(i)}(\varepsilon_1, \varepsilon_2; \Lambda') = F^{(i)}(\varepsilon_1, \varepsilon_2; \Lambda)$, showing (6).

Next, see figure 4. The right-hand side of the figure represents the left-hand side of (9), the bullet • showing the summation $\sum_{\varepsilon=\pm}$. Using unitarity, as shown on the left-hand

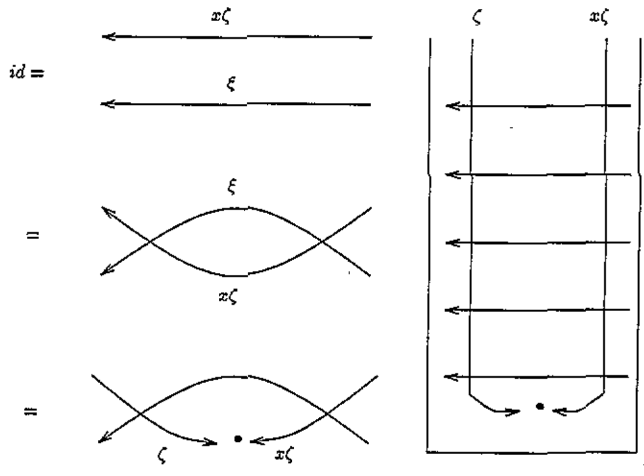


Figure 4. The unitarity with the two horizontal lines oriented left, and the case where the orientation of the left half of one of the lines is reversed.

side of the figure, we can push away the two vertical lines with the spectral parameters ζ and $x\zeta$. Therefore, we have (9).

Equation (10) follows from (6) and (3); (7) is obvious; (8) follows from the parity of the Boltzmann weights.

Finally, we will show (11)–(12). Since the arguments are the same, we will consider only (11). See figure 5, where ζ_1' in (11) is shown as ζ. We can push away the semi-circle with the spectral parameter ζ. In the limit, we have (11).

3. Spontaneous staggered polarization

Let us work out the case $n = 2$ in detail. Since $F_2^{(i)}(\zeta_1, \zeta_2)$ depends only on the ratio $\zeta = \zeta_1/\zeta_2$, we shall denote $F_2^{(i)}(\zeta_1, \zeta_2)$ by $F_2^{(i)}(\zeta)$. Set $G^{\pm}(\zeta) = F_2^{(0)}(\zeta) \pm F_2^{(1)}(\zeta)$. Equation (5) for $F_2^{(i)}(\zeta)$ reads $G^{\pm}(x^{-2}\zeta) = \pm R(\zeta)G^{\pm}(\zeta)$. Using (7), (8) we have $G^{\pm}_{+-}(\zeta) = \pm G^{\pm}_{-+}(\zeta)$, $G^{\pm}_{+-}(-\zeta) = G^{\mp}_{+-}(\zeta)$. Thus the equations reduce to

$$G^{+}_{+-}(x^{-2}\zeta) = \bar{\kappa}(\zeta^2)^{-1}\frac{(x\zeta^{-1};q)_\infty(qx^{-1}\zeta;q)_\infty}{(x\zeta;q)_\infty(qx^{-1}\zeta^{-1};q)_\infty}G^{+}_{+-}(\zeta)\,. \tag{14}$$

Set

$$\varphi(z) = g(z)g(x^{-4}z^{-1}) \qquad g(z) = \frac{(x^6z;q,x^4,x^4)_\infty(qx^6z;q,x^4,x^4)_\infty}{(x^4z^{-1};q,x^4,x^4)_\infty(qz^{-1};q,x^4,x^4)_\infty}$$

which solves the equation $\varphi(x^{-4}z) = \bar{\kappa}(z)^{-1}\varphi(z)$. It is easy to see that the following is a solution of (14) satisfying the normalization (9), i.e. $G^{+}_{+-}(x^{-1}) = 1$:

$$G^{+}_{+-}(\zeta) = \frac{\varphi(\zeta^2)}{\varphi(x^{-2})}\frac{(x^2;x^2)^2_\infty}{(q;q)^2_\infty}\frac{(qx\zeta;q)_\infty(qx^{-1}\zeta^{-1};q)_\infty}{(x^3\zeta;x^2)_\infty(x\zeta^{-1};x^2)_\infty}\,. \tag{15}$$

Moreover the solution is unique if we impose further the condition that $G^{+}_{+-}(\zeta)$ is holomorphic in the neighbourhood of $x \leqslant |\zeta| \leqslant x^{-1}$. In view of (4) and following

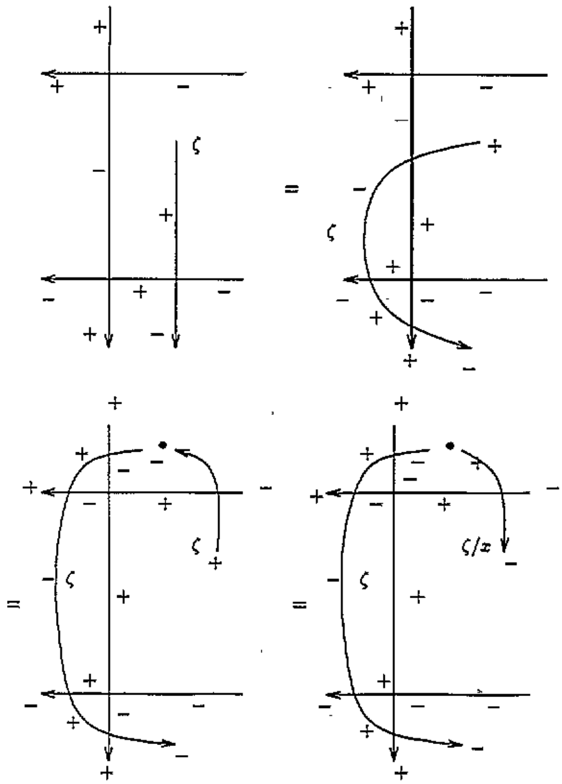


Figure 5. The use of the Yang–Baxter equation to rotate a half-line. A choice of GSC is written explicitly. Notice that after rotating the GSC changes.

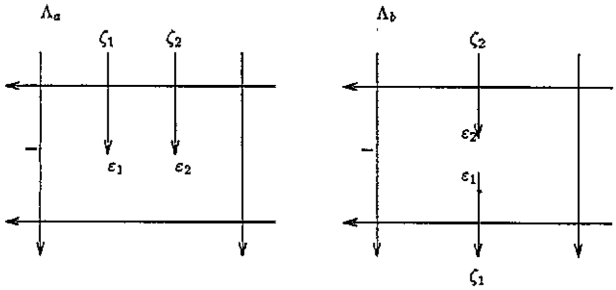


Figure 6. Two types of dislocation. The minus sign shows the $i = 0$ GSC.

the definition given in [9], the staggered polarization P_0 is

$$P_0 = \frac{G^{-}_{+-}(x^{-1})}{G^{+}_{+-}(x^{-1})} = \frac{(-q;q)^2_\infty (x^2;x^2)^2_\infty}{(q;q)^2_\infty (-x^2;x^2)^2_\infty}.$$

This formula was conjectured by Baxter and Kelland [6]. It was proved in the trigonometric case $q = 0$ by Baxter [9]. Our result is stronger than this: in the context of this section, it says that the ratio of the configuration sums for the lattice Λ_a in figure 6, in the $i = 0$ sector, with the two different choices of the spins at the end edges, i.e. $(\varepsilon_1, \varepsilon_2) = (\pm, \mp)$, is given by

$$\frac{1}{2}\frac{G^+_{+-}(\zeta)+G^-_{+-}(\zeta)}{G^+_{-+}(\zeta)-G^-_{-+}(\zeta)}$$

and that of the lattice Λ_b is given by

$$\frac{1}{2}\frac{G^+_{+-}(\zeta/x)+G^-_{+-}(\zeta/x)}{G^+_{-+}(\zeta/x)-G^-_{-+}(\zeta/x)}.$$

We have checked these statements to a first few orders in the low-temperature expansions.

4. Vertex operators

In this section we shall reformulate the construction of section 1 as an operator theory. To this end let us first recall the corner transfer matrices (CTMs) following [7]. Consider the lattice drawn in figure 7. Number the rows (resp. columns) from bottom to top (resp. from right to left) as $-N+1,\ldots,0,1,\ldots,N$. Fixing the boundary spins to the ith GSC, we denote by $A_0(\zeta)$, $A_1(\zeta)$, $A_2(\zeta)$, $A_3(\zeta)$ the CTM corresponding to the NE, SE, SW, NW quadrant respectively. When formulated in the IRF language they are the transpose of B, A, D, C in [7, p 366]. For instance the (σ,σ') element of $A_3(\zeta)$ is the partition function of the NW quadrant where the horizontal (resp. vertical) boundary spins are fixed to $\sigma=(\varepsilon_1,\ldots,\varepsilon_N)$ (resp. $\sigma'=(\varepsilon'_1,\ldots,\varepsilon'_N)$). From the relations (2) it follows that $A_2(\zeta)=\mathcal{R}A_3(x^{-1}\zeta^{-1})$, $A_1(\zeta)=\mathcal{R}A_3(\zeta)\mathcal{R}$, $A_0(\zeta)=A_3(x^{-1}\zeta^{-1})\mathcal{R}$, where $\mathcal{R}=\sigma^x\otimes\cdots\otimes\sigma^x$ denotes the spin reversal operator. Normalize the CTMs so that the largest eigenvalue of $A_3(\zeta)$ is 1. Baxter's argument shows that in the principal regime we have $\lim_{N\to\infty}A_3(\zeta)=\zeta^{-D}$, where the operator D is independent of ζ and has discrete eigenvalues $0,1,2,\ldots$. Consequently

$$\lim_{N\to\infty}A_0(\zeta)A_1(\zeta)A_2(\zeta)A_3(\zeta)=x^{2D}. \tag{16}$$

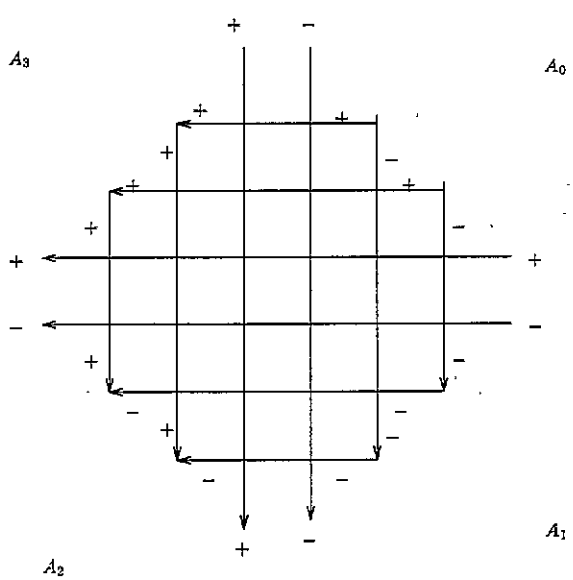


Figure 7. The CTMs $A_0,\ldots,A_3$. The 0th GSC is depicted.

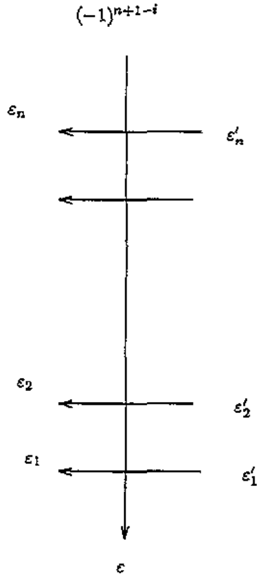


Figure 8. Matrix element $\left(\Phi^{i+1}_{i\,\varepsilon}(\zeta)\right)_{\sigma\sigma'}$ where $\sigma = (\varepsilon_1, \ldots, \varepsilon_n)$, $\sigma' = (\varepsilon'_1, \ldots, \varepsilon'_n)$.

Denote by $\mathcal{H}_i$ the space of eigenvectors of $A_3(\zeta)$ in the limit. In the trigonometric case $q = 0$, as was shown in [10, 1], $\mathcal{H}_i$ can be identified with the level-1 highest-weight module $V(\Lambda_i)$ of the quantized affine algebra $U_{-x}(\hat{sl})$ (with $-x$ playing the role of the deformation parameter), and D acts as the grading operator in the prinicipal gradation. In the elliptic case, we have no such representation theoretical picture. However if $\mathcal{H}_i = \oplus_{d=0}^{\infty}\mathcal{H}_{i,d}$ is the eigenspace decomposition for D, then $\dim \mathcal{H}_{i,d}$ cannot change continuously. Hence it should be the same as the $q = 0$ case, giving

$$\mathrm{tr}_{\mathcal{H}_i} x^{2D} = \sum_{d=0}^{\infty} x^{2d} \dim \mathcal{H}_{i,d} = \frac{1}{(x^2; x^4)_\infty}. \tag{17}$$

Consider now the operator whose (σ, σ') matrix element is given by figure 8. We expect that in the large-lattice limit $N \to \infty$ it will give rise to a well defined operator

$$\Phi^{i+1}_{i\,\varepsilon}(\zeta) \ : \ \mathcal{H}_i \longrightarrow \mathcal{H}_{i+1}. \tag{18}$$

The effect of operating with $\Phi^{i+1}_{i\,\varepsilon}(\zeta)$ amounts to inserting a vertical dislocation in the lattice. In view of the expression (16) of the CTMs the function $F^{(i)}_n$ of section 1 is expressed as

$$F^{(i)}_n(\zeta_1, \ldots, \zeta_n)_{\varepsilon_1,\ldots,\varepsilon_n} = \frac{\mathrm{tr}_{\mathcal{H}_i}\left(x^{2D}\Phi^{i}_{i+1\,\varepsilon_1}(\zeta_1)\cdots\Phi^{i+n-1}_{i+n\,\varepsilon_n}(\zeta_n)\right)}{\mathrm{tr}_{\mathcal{H}_i}\left(x^{2D}\right)}. \tag{19}$$

The arguments in section 1 can be translated to the properties of $\Phi^{i+1}_{i\,\varepsilon}(\zeta)$. We summarize them below:

$$x^{2D} \circ \Phi^{i+1}_{i\,\varepsilon}(\zeta) \circ x^{-2D} = \Phi^{i+1}_{i\,\varepsilon}(x^2\zeta) \tag{20}$$

$$\Phi^{i}_{i+1\,\varepsilon_2}(\zeta_2)\Phi^{i+1}_{i\,\varepsilon_1}(\zeta_1) = \sum_{\varepsilon'_1,\varepsilon'_2} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2;\varepsilon'_1\varepsilon'_2}\Phi^{i}_{i+1\,\varepsilon'_1}(\zeta_1)\Phi^{i+1}_{i\,\varepsilon'_2}(\zeta_2). \tag{21}$$

There exists a linear isomorphism $\nu : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ such that

$$\nu \circ D = D \circ \nu \qquad \Phi^1_{0\varepsilon}(\zeta) = \nu \circ \Phi^0_{1-\varepsilon}(\zeta) \circ \nu \tag{22}$$

$$\Phi^{i+1}_{i\varepsilon}(-\zeta) = (-1)^{i+1}\varepsilon\, \Phi^{i+1}_{i\varepsilon}(\zeta) \tag{23}$$

$$\sum_{\varepsilon=\pm} \Phi^{*i}_{i+1\,\varepsilon}(\zeta)\Phi^{i+1}_{i\varepsilon}(\zeta) = \mathrm{id} \qquad \Phi^{i+1}_{i\varepsilon}(\zeta)\Phi^{*i}_{i+1\,\varepsilon'}(\zeta) = \delta_{\varepsilon\varepsilon'} \times \mathrm{id}\,. \tag{24}$$

In (24) we have set $\Phi^{*i+1}_{i\varepsilon}(\zeta) = \Phi^{i+1}_{i-\varepsilon}(x^{-1}\zeta)$. The difference equations (5) for (19) are an immediate consequence of (20), (21). In the trigonometric case (18) is related to the vertex operators in [1,2] via $\Phi^{i+1}_{i\varepsilon}(\zeta) = g^{-1/2}\zeta^{(1+\varepsilon)/2-i}\widetilde{\Phi}^{\Lambda_{i+1}}_{\Lambda_i\varepsilon}(\zeta^2)$ where g is a scalar independent of ζ.

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